

Final Adventure

Math 3310

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Rules of Engagement

You may discuss the problems below with anyone in the following set of people:

$$P = \{\text{Anyone currently enrolled in MATH 3310,} \\ \text{Heidi Anderson, Ella Leonelli, Elsa Schutfort, Erin Pitts, Brent Thomas}\}.$$

The documentation of your adventure should be created by teams of at most four and are to be submitted by Wednesday May 1, 2024 as a pdf via Canvas (the way we've done all semester). One submission per team.

No outside resources are to be used on this **Experience**. Try to get through this with your brain and the brains of your team (if you choose to be part of one).

1 Preparing for a Math Party

During your last adventure you had so much fun playing games that you have decided to find new ways to play games with the help of some math. And of course you decide to organize another game night. Since you already have a variety of games that use dice, you set out to find a way to make the dice more interesting.

Funky Dice

Sub-Adventure 1.1

A 6-sided die labeled with the integers 1, 2, 3, 4, 5, 6 will be called a *standard die*. The goal for this part of the *Final* Adventure is to determine all ways to label a pair of dice with positive integers so that the probabilities of rolling the usual sums 2, 3, ..., 12 are the same, but the labels are non-standard.

- (a) Let $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$, and explain why $(p(x))^2$ is the generating function for the probabilities of outcomes in rolling a pair of standard dice.

Solution:

$p(x)$ is the generating function for rolling 1 standard die. This is because there are 6 possible outcomes of rolling a standard die: 1, 2, 3, 4, 5, and 6. Each outcome is equally likely, and there is only one way to roll each number.

We enumerate the number of ways to achieve each possible outcome of rolling two standard dice:

# Rolled	# of Ways
1	0
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

The number of ways the dice can be rolled are the coefficients for the generating function. This means the generating function should be

$$x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$$

This is equal to $(p(x))^2$.

Therefore, $(p(x))^2$ is the generating function for rolling a pair of standard dice.

- (b) Let $A = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $B = (b_1, b_2, b_3, b_4, b_5, b_6)$ be two lists of positive integers. Put $p_A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$ and $p_B(x) = x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}$. Explain why finding a_i s and b_i s such that $p_A(x)p_B(x) = (p(x))^2$ is relevant this part of the Adventure.

Solution:

$(p(x))^2$ is the generating function for rolling a pair of standard dice.

We want to create a pair of nonstandard dice that act like a pair of standard dice. So they should have the same generating function as a pair of standard dice.

Therefore, multiplying the generating function of each non-standard die should yield the generating function $(p(x))^2$ in order to have the same probabilities as a pair of standard dice.

- (c) Factor $p(x)$ into irreducible polynomials and use this factorization to help solve for the a_i s and b_i s. Specifically, the factorization will force the form of $p_A(x)$ to be something like $p_1(x)^q p_2(x)^r p_3(x)^s p_4(x)^t$, where $0 \leq q, r, s, t \leq 2$ and $p_i(x)$, for $1 \leq i \leq 4$, is a factor of $p(x)$. In your solution to this step, you must motivate why you take this step.

Solution:

Start with the function $p(x)$

$$p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$$

Then, factor out an x .

$$p(x) = x(x^5 + x^4 + x^3 + x^2 + x + 1)$$

Split the non-factored term into two pieces.

$$p(x) = x((x^5 + x^3 + x) + (x^4 + x^2 + 1))$$

Factor out an x and a 1 to make the terms in parentheses match.

$$p(x) = x(x(x^4 + x^2 + 1) + 1(x^4 + x^2 + 1))$$

Now factor out the $x + 1$.

$$p(x) = x(x + 1)(x^4 + x^2 + 1)$$

Add and subtract an x^3 , x^2 , and x and split up the non-factored terms.

$$p(x) = x(x + 1)((x^4 - x^3 + x^2) + (x^3 - x^2 + x) + (x^2 - x + 1))$$

Pull out an x^2 , x , and 1 for each term respectively to make the terms in parentheses match.

$$p(x) = x(x + 1)(x^2(x^2 - x + 1) + x(x^2 - x + 1) + 1(x^2 - x + 1))$$

Now factor out the $x^2 - x + 1$.

$$p(x) = x(x + 1)(x^2 - x + 1)(x^2 + x + 1)$$

Now we have a factorization that looks like the form of

$$p(x) = p_1(x)p_2(x)p_3(x)p_4(x)$$

So we'll throw in the labels given by the question and we get that our generating function $p_A(x)$ is in the form of

$$p_A(x) = x^q(x + 1)^r(x^2 - x + 1)^s(x^2 + x + 1)^t$$

- (d) Begin to reduce the possibilities for q, r, s , and t by using information from $p_A(1)$ and $p_A(0)$. Note that, on one hand $p_A(1) = 1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6} = 6$ (since $a_i > 0$), and on the other hand we have $p_A(1) = p_1(1)^q p_2(1)^r p_3(1)^s p_4(1)^t$. Similarly, there are two ways to view $p_A(0)$.

Solution:

We know that $p_A(0) = 0$, because $a_i > 0$. This means there must be an x term, otherwise $p_A(0)$ would equal 1. So $1 \leq q \leq 2$ in order for there to be an x term.

However, because the other die should also have $p_B(0) = 0$, the other die must also have an x term. Therefore, $q = 1$.

Additionally, because $p_A(0) = 0$, there cannot be a 0 term on the die at all. Otherwise, $p_A(0)$ would have to be ≥ 1 .

Finally, because $p_A(1) = 6$, the die has to be 6-sided. Otherwise the 1^{a_i} terms would not add up to 6.

- (e) List all possible ways to label a pair of dice so that the probabilities of obtaining the sums 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$, respectively. One such way will be the standard way. In your solution for this step, explain why you have proved that the labels you have found are the only possible ones that give the desired probabilities for roll-outcomes.

Solution:

Now, we can use the information listed above to find the dice that meet the requirements.

Let $Q = x$, $R = x + 1$, $S = x^2 - x + 1$, and $T = x^2 + x + 1$.

Using these terms, we will list all the possible dice pairs that meet the requirement of containing 1 Q. In each case, there must not be more than 2 instances of R, S, or T in order to meet the requirement of $0 \leq q, r, s, t \leq 2$.

Below we list both $p_A(x)$ and $p_B(x)$ to avoid computing duplicates. Each $p_A(x)$ has a $p_B(x)$ pair. So if a setup appears on the $p_B(x)$ side, we do not need to check if it is valid because we already check that dice pair.

	$p_A(x)$	$p_B(x)$	$p_A(x)$ Expanded
1	Q	QRRSSTT	x
2	QR	QRSSTT	$x + x^2$
3	QRR	QSSTT	$x + 2x^2 + x^3$
4	QRS	QRSTT	$x + x^3$
5	QSS	QRRTT	$x - 2x^2 + 3x^3 - 2x^4 + x^5$
6	QRRS	QSTT	$x + x^2 + x^4 + x^5$
7	QRSS	QRTT	$x - x^2 + x^3 + x^4 - x^5 + x^6$
8	QRST	QRST	$x + x^2 + x^3 + x^4 + x^5 + x^6$
9	QSST	QRRT	$x - x^2 + 2x^3 - x^4 + 2x^5 - x^6 + x^7$
10	QRRSS	QTT	$x + 2x^4 + x^7$
11	QRRST	QST	$x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + x^7$
12	QRSST	QRT	$x + x^3 + x^4 + x^5 + x^6 + x^8$
13	QRRSST	QT	$x + x^2 + x^3 + 2x^4 + 2x^5 + 2x^6 + x^7 + x^8 + x^9$
14	QRRSTT	QS	$x + 3x^2 + 5x^3 + 6x^4 + 6x^5 + 6x^6 + 5x^7 + 3x^8 + x^9$

Now that we have a list of all the dice, we can remove the die that do not fit the requirements. First, dice with negative values do not make sense in the context of this problem (as no die can exist with a negative number of faces), so we will remove them. Second, we found that the dice must contain 6 sides, so we will remove dice that have more or less sides.

After doing this, we are left with 2 pairs of dice. These pairs of dice are option 8 and option 12.

The build for option 8 is

$$\begin{aligned}
 p_A(x) &= QRST \\
 p_A(x) &= x(x+1)(x^2-x+1)(x^2+x+1) \\
 p_A(x) &= x + x^2 + x^3 + x^4 + x^5 + x^6
 \end{aligned}$$

This is just a standard pair of dice. So option 8 is the set of standard dice.

The build for option 12 is

$$p_A(x) = QRSST$$

$$p_A(x) = x(x+1)(x^2-x+1)(x^2-x+1)(x^2+x+1)$$

$$p_A(x) = x + x^3 + x^4 + x^5 + x^6 + x^8$$

and

$$p_B(x) = QRT$$

$$p_B(x) = x(x+1)(x^2+x+1)$$

$$p_B(x) = x + 2x^2 + 2x^3 + x^4$$

That means the only other possible set of dice that meet the requirements is a pair of 6-sided dice where the first die has the sides $[1, 3, 4, 5, 6, 8]$ and the second die has the sides $[1, 2, 2, 3, 3, 4]$.

Mastering “Eeny, meeny, miney, moe”

Sub-Adventure 1.2

As you continue preparing for your game night with your friends, you decide that you need a way to choose which player will go first. Of course, the first thing you think about to choose someone fairly is the classic selection strategy “Eeny, meeny, miney, moe.” You decide that you will put all of your friends in a circle and number them 1 to n . Starting with the friend number 1 you will go around removing every other person from the circle until only one person is left. The last person in the circle will be the first player. To gain a competitive advantage you decide that you will find a formula, $S(n)$, to describe the position that will be last in the circle if there are n people in the circle to start. That way you will be able to make yourself first player without anyone realizing what you are doing. You begin by calculating $S(4) = 1$ and $S(7) = 7$. Give a formula for $S(n)$.

Solution:

Lemma 1: $S(2^k) = 1$ for $k \in \mathbb{Z}^{\geq 0}$

Proof for Lemma 1:

Start with a circle of 2^k people and let $n = 2^k$. For ease of understanding, we will relabel people to have the first person labeled 0 and the last person labeled $n - 1$.

We will begin by going around the circle one time. As we go around, we remove every ‘odd’ labeled person. We begin by skipping over player 0, removing player 1, skipping player 2, removing player 3, and so on.

As we get to the end of the circle, we end when player $n - 1$ is removed. This effectively gives us a circle that is half the size of the one we started with. Because we started with a circle of base 2, every time we go around the circle eliminating half of the people, we are left with a circle half the size. Now, player 1 is the person able to make the first move the next time around the circle.

At this point, all of the odd people are removed from the group, and there are 2^{k-1} people remaining. Again, it should be noted that circle of remaining players is a multiple of 2.

Before we go through the circle again, for ease of following, we will renumber the remaining people. Because everyone left is an even number, we can take their place in the circle and divide it by 2. Again, we have a circle of size 2^{k-1} . So, after renumbering, everyone will have a number from 0 to 2^{k-1} incrementing by 1.

We go through the circle again. This time we start with player 0 removing player 1, player 3, player 5, and so on. This round ends when player $n - 2$ is removed. Once again, player 0 will be the starting position of the next round. All the of people who are not a multiple of 4 (2^2) have been removed and there are 2^{k-2} players remaining.

This pattern continues until there are two people left. Because we always start with person 0 when handling a 2^k sized circle, person 1 is eliminated and person 0 wins. This shows that no matter what size of 2^k circle we have, player 0 always wins. Thus $S(2^k) = 1$ for $k \in \mathbb{Z}^{\geq 0}$.

Our circle of friends is not limited to size 2^k . If it were we could just sit at the number one position every time to win. But, we are always able to make a circle of size 2^k from our friends. We will always have a circle of friends that is size $2^k + R$, where R are the friends ruining our idea to sit at position 0 to always win.

Our first time going around the circle we will always eliminate the R players. Immediately after R players are eliminated there is a circle of size 2^k remaining. Now that R players are eliminated the person who we've landed on will win. This is due to Lemma 1. The person landed on after R players are eliminated is person 0 of a newly created 2^k sized circle.

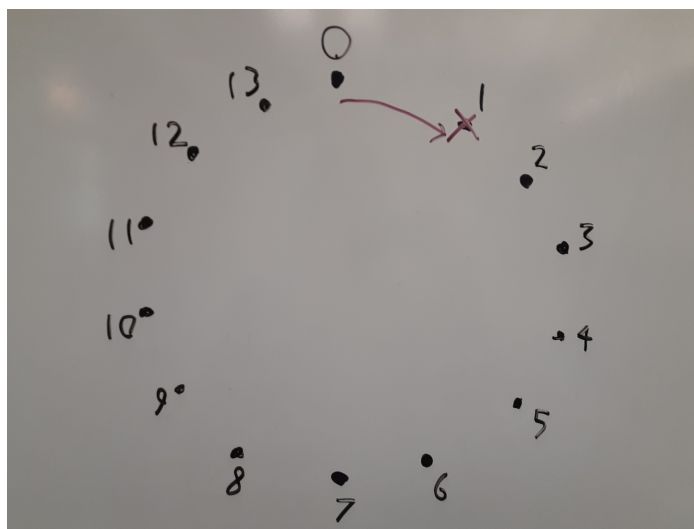


Figure 1. Visualization of scenario with 14 players.

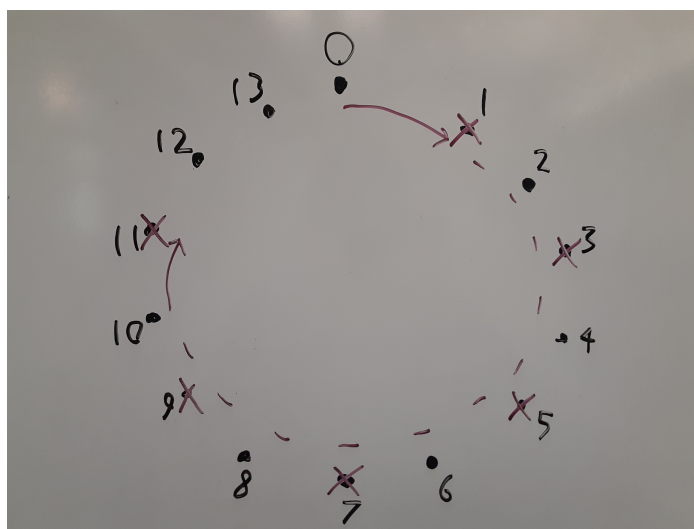


Figure 2. Removal of R players leaves 2^k players remaining. In this case, $R = 6$ and $k = 3$.

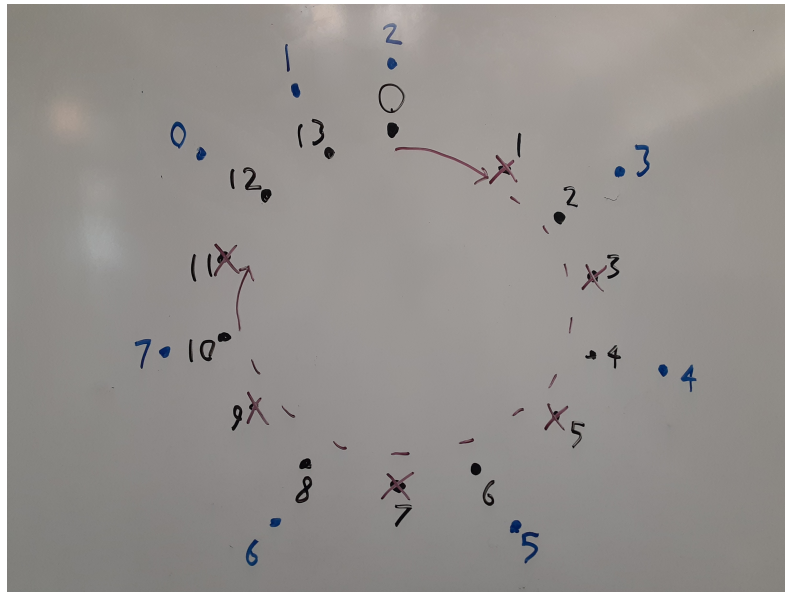


Figure 3. Remaining players can be reindexed as a ring of 2^k players, with the first player after R eliminations being player 0.

We want to be the +1 after R players are eliminated. We multiply R by 2 since we eliminate every other person, thus there are 2 people in the circle for every 1 person of R eliminated. The person who wins and where we should sit is

$$S(n) = 2(R) + 1, \quad n = 2^k + R$$

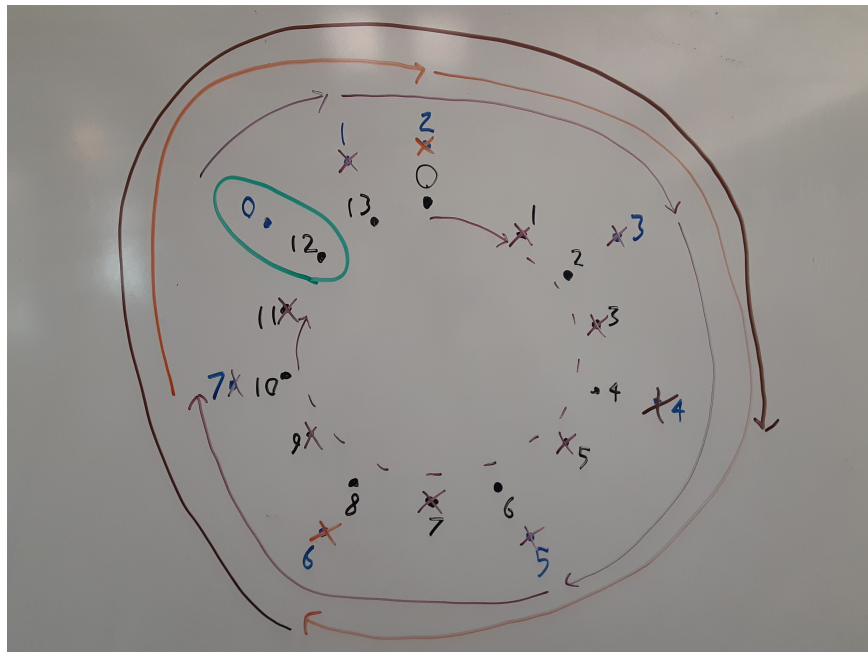


Figure 4. Game is won by player $2(R) + 1$. In this case, $2(R) - 1 = 13$ (12 in reindexed paradigm).

2 Invitation Madness

Sub-Adventure 2

You decide to use snail mail to send out invitations to the game night. Since you have never sent a piece of paper mail before in your life you start to worry that no one will receive the correct invitation. It occurs to you that there are more ways that every one you invited to receive an invitation addressed to someone else than there are for everyone to receive the correct invitation. This only serves to exacerbate your anxiety. You decide to calculate the number of ways that all of the invitations can be sent to the wrong person.

Determine the number D_n of ways the n invitation can be delivered to your n friends so that no one gets the invitation actually intended for them.

Moreover, it is asked that you determine D_n in a few ways: one way entails finding a recurrence relation for D_n , another entails a closed (albeit complicated) formula for D_n , and finally a fairly simple formula for D_n obtained from the more complicated one.

(a) Prove the recurrence $D_n = (n - 1)D_{n-1} + (n - 1)D_{n-2}$, for $n \geq 2$, with $D_0 = 1$ and $D_1 = 0$,

Solution:

You have n invitations that you need to figure out all the different ways you can send wrong. Start by singling out invitation 1. This invitation cannot go to person 1, otherwise it would be sent correctly. That gives us $n - 1$ people to whom invitation 1 can be sent.

Now let invitation 1 be sent to person m , where $m \in \mathbb{Z}$ and $1 \leq m \leq n$.

There are 2 options for what can happen now:

1. Invitation m is sent to person 1.

In this case, invitation 1 and invitation m are sent to the wrong recipients, and now we just have the remaining $n - 2$ invitations to send out incorrectly.

By definition, this can be done in D_{n-2} different ways.

So in this case, there are $(n - 1)D_{n-2}$ ways to send out the wrong invitations.

2. Invitation m is not sent to person 1.

In this case, we have one restriction, invitation m cannot be sent to the first person, otherwise we would have our first case.

In this case, there are $n - 1$ invitations that need to be sent to $n - 1$ people and each person has one invitation they cannot receive.

That is the same as D_{n-1} .

So in this case, there are $(n - 1)D_{n-1}$ ways to send out the invitations incorrectly.

If we add these two cases together, we get the equation

$$D_n = (n - 1)D_{n-1} + (n - 1)D_{n-2}$$

This is the recurrence relation we were trying to prove.

- (b) Derive, from the recurrence in sub-adventure 2(a), this recurrence $D_n = nD_{n-1} + (-1)^n$, for $n \geq 1$, with $D_0 = 1$.

Solution:

Begin with the equation

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

Expand the first term of the right side.

$$D_n = nD_{n-1} - D_{n-1} + (n-1)D_{n-2}$$

Now, subtract nD_{n-1} from both sides.

$$D_n - nD_{n-1} = -D_{n-1} + (n-1)D_{n-2}$$

Pull out a negative on the right side.

$$D_n - nD_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2})$$

Now, the right side of the equation is the same as the left side, just with a negative and down one index. So we can keep lowering the index by one, if we just multiply by a negative each time.

The first time going down, that gives us

$$D_n - nD_{n-1} = (-1)^2(D_{n-2} - (n-2)D_{n-3})$$

Keep going for

$$D_n - nD_{n-1} = (-1)^3(D_{n-3} - (n-3)D_{n-4})$$

We can keep reducing the index until eventually, we get to this point

$$D_n - nD_{n-1} = (-1)^{n-1}(D_1 - (n - (n-1))D_0)$$

Simplify

$$D_n - nD_{n-1} = (-1)^{n-1}(D_1 - D_0)$$

We know that $D_1 = 0$ and $D_0 = 1$, so we will plug those in.

$$D_n - nD_{n-1} = (-1)^{n-1}(0 - 1)$$

Simplify

$$D_n - nD_{n-1} = -(-1)^{n-1}$$

Pull out the -1 from the exponent

$$D_n - nD_{n-1} = \frac{-(-1)^n}{-1}$$

Simplify

$$D_n - nD_{n-1} = (-1)^n$$

Now, we just need to add the nD_{n-1} back.

$$D_n = nD_{n-1} + (-1)^n$$

And now we have the equation we were trying to derive.

Therefore,

$$D_n = nD_{n-1} + (-1)^n$$

is equivalent to

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

- (c) Use the Principle of Inclusion-Exclusion to compute a closed formula for D_n .

Solution:

The Principle of Inclusion/Exclusion: Suppose $A_1, A_2, \dots, A_k$ are subsets of U . Then

$$|\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}| = |U| - \sum_{i=1}^k |A_i| + \sum_{i \neq j} |A_i \cap A_j| - \dots + (-1)^k |A_1 \cap A_2 \cap \dots \cap A_k|$$

We desire to get D_n . This can be achieved using the formula given in class.

$$D_n = |U| + \sum_{k=1}^n (-1)^k \binom{n}{k} f(k)$$

It is now up to us to define the terms and simplify algebraically.

First we will look at $|U|$. The $|U|$ is determined by how many ways there are to send a letter to everyone. In our case the number of people receiving letters is n . The number of ways we can send n letters to n people where each person and letter is unique is $n!$.

$$|U| = n!$$

Let A_i be the event that person i gets the correct invitation. A_1 would be the event that friend 1 gets the correct invitation. $A_1 \cap A_2$ would be the event that both friend 1 and 2 gets the correct invitation. $A_1 \cap A_2 \cap \dots \cap A_n$ would be the event that all n friends receive the correct invitation.

When 1 friend gets the correct letter, it removes a letter from potentially being sent to any of the other friends. The other letters can still be sent in any combination as long as they follow our delivery rules.

$$A_1 = (n-1)! = f(1)$$

When 2 friends get the correct letter, it removes two letters from being sent to other friends. The other letters can still be sent in any combination.

$$A_1 \cap A_2 = (n-2)! = f(2)$$

This trend follows for k friends.

$$A_1 \cap A_2 \cap \dots \cap A_k = (n-k)! = f(k)$$

Now that we have values for $|U|$ and $f(k)$ we can plug them into are D_n

$$D_n = n! + \sum_{k=1}^n (-1)^k \binom{n}{k} (n-k)!$$

Expanding out the choose function allows us to cancel some terms

$$D_n = n! + \sum_{k=1}^n (-1)^k \frac{n!}{k!(n-k)!} (n-k)!$$

$$D_n = n! + \sum_{k=1}^n \frac{(-1)^k n!}{k!}$$

We can multiply $n!$ by 1 to get a more convenient form

$$D_n = n! \left(\frac{(-1)^0}{0!} \right) + \sum_{k=1}^n \frac{(-1)^k n!}{k!}$$

The first addend is the 0^{th} term of the sum from the second addend. Moving the first into the sum gives us the final equation

$$D_n = \sum_{k=0}^n \frac{(-1)^k n!}{k!}$$

(d) Noting that e^x has a power series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots,$$

prove that D_n is the nearest integer to $\frac{n!}{e}$; that is, show that

$$D_n = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor.$$

Solution:

Lemma 2:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Proof: Left to reader.

This is what we got for D_n :

$$D_n = \sum_{k=0}^n \frac{(-1)^k n!}{k!}$$

For this problem we will pull out $n!$ since it is a constant.

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$$

Notice that this looks similar to a power series expansion. We can rewrite D_n as

$$D_n = n! \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right)$$

By Lemma 2, the first sum is just the power series expansion for e^{-1} , so we get

$$D_n = (n!) \frac{1}{e} - (n!) \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}$$

Notice that the remaining sum represents the error and will be largest when $n = 1$ (only considering $n \geq 1$). When $n = 1$,

$$(1!) \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} \approx 0.36788$$

Thus, the error at its largest can be accounted for by simply rounding to the nearest integer. Take our first term - the approximation - and account for the error by adding $1/2$ and rounding down. This ensures that the approximation is rounded to the nearest integer. This gives the formula

$$D_n = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor$$

(Now that's a nice and concise closed formula!)

3 PIE

Once you have the invitation debacle straightened out and you know how to rig the games so you can be first player, you are chatting with a friend about the game night. She mentions that you should think about pie for the party. Since you just solved the invitation madness problem using the P.I.E. you didn't realize she was talking about the food and you thought she was talking about the counting technique. So you "think about pie" by solving some more problems using the P.I.E.

Sub-Adventure 3.1

Use the Principle of Inclusion/Exclusion to determine a formula for $k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. (Hint: you should try counting $k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$.)

Solution:

We'll start by counting $k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. This is the number of ways to order n labeled balls into k labeled boxes with none of the boxes empty and was proved in Homework 9. We'll count this another way using the Principle of Inclusion/Exclusion.

The Principle of Inclusion/Exclusion: Suppose $A_1, A_2, \dots, A_k$ are subsets of U . Then

$$|\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}| = |U| - \sum_{i=1}^k |A_i| + \sum_{i \neq j} |A_i \cap A_j| + \dots + (-1)^k |A_1 \cap A_2 \cap \dots \cap A_k|$$

To count $k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$, we'll let A_i be the event that the i th box is empty. Thus, $|\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}|$ describes the scenario that every box is non-empty and is equal to $k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. In this case U is the number of ways to order labeled balls into labeled boxes with no restrictions. So, $|U| = k^n$, as proven in Homework 10.

$|A_i|$ is the number of ways to leave the i th box empty. If only one box is empty, then there are $k-1$ boxes left to order n balls. Of the k boxes, we need to choose 1 of them to be empty. Therefore, the number of ways to order n balls into k boxes with one box empty is $\binom{k}{1}(k-1)^n$.

$|A_i \cap A_j|$ where $i \neq j$ is the number of ways to leave 2 boxes empty. First choose which boxes are empty - $\binom{k}{2}$. This gives us the number of ways to order n balls into k boxes with 2 empty ones is

$$\binom{k}{2}(k-2)^n.$$

At this point we can notice the pattern and put all the pieces together using PIE.

$$k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = |\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}| = k^n + \sum_{i=1}^k (-1)^i \binom{k}{i} (k-i)^n$$

Notice that k^n is just the $i = 0$ term in the sum, so we can absorb it into the sum by letting i go from 0 to k .

$$k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$$

We can also rewrite the choose function as $\frac{k!}{i!(k-i)!}$. This gives us

$$k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{i=0}^k (-1)^i \frac{k!}{i!(k-i)!} (k-i)^n$$

To find a formula for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ from here, all we need to do is divide by $k!$ to get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{i=0}^k (-1)^i \frac{k!}{i!(k-i)!k!} (k-i)^n$$

Simplify one more time to get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{i=0}^k \frac{(-1)^i (k-i)^n}{i!(k-i)!}$$

Sub-Adventure 3.2

If you are randomly dealt 10 cards from a standard deck¹ then what is the probability that you have at least 1 four of a kind²?

Solution:

Let x be an event where you draw 10 random cards and have at least one four of a kind.

The probability of this event occurring is

$$p(x) = \frac{\binom{13}{1}\binom{48}{6} - \binom{13}{2}\binom{44}{2}}{\binom{52}{10}} \approx 0.01$$

Now to break this down a bit:

The denominator is the number of total possible ways to draw 10 cards from a deck of 52 cards.

The numerator is the number of possible ways event x can occur.

This works because there are $\binom{13}{1}$ possible ways to select one of the 13 types of card to be the four of a kind and $\binom{48}{6}$ ways to pick the remaining 6 cards for your hand from the remaining 48 cards of the deck.

This, however, has an issue of double counting. For example, if we choose the kings in $\binom{13}{1}$, there is a scenario in which we pull four aces when picking the remaining 6 cards. However, the same scenario is counted when we select the aces to be our four of a kind.

To compensate for this double counting that occurs when two four of a kind are pulled, we just need to subtract one copy.

We do this by selecting all the ways to pick 2 types of cards. That's $\binom{13}{2}$ and multiply this by the number of ways to pick the remaining 2 cards. That's $\binom{44}{2}$.

So altogether, $\binom{13}{1}\binom{48}{6} - \binom{13}{2}\binom{44}{2}$ is the number of ways in which event x can occur.

Therefore, to get the probability of event x happening, we just need to divide $\binom{13}{1}\binom{48}{6} - \binom{13}{2}\binom{44}{2}$ by all possible ways to draw 10 cards.

¹A standard deck of card consists of 52 cards in each of four different suits of Spades, Hearts, Diamonds, and Clubs. Each suit contains 13 cards: Ace, 2,3,4,5,6,7,8,9,10, Jack, Queen, King.

²Four cards of the same value from each of the different suits.